Quantum Interference In Computer Vision

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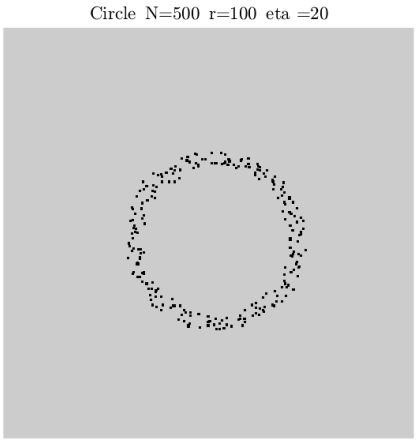
Publications

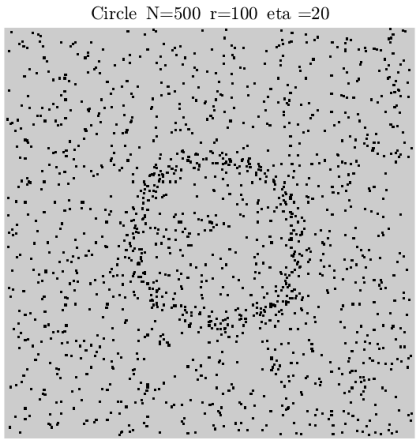
1. *Complex-Valued Hough Transforms for Circles.* M. Cicconet, D. Geiger and M. Werman. In the IEEE 22nd International Conference in Image Processing (ICIP), 2015, Quebec City, Canada.
2. *Quantum Interference and Shape Detection*. D. Geiger and Z. Kedem. Energy Minimization Methods in Computer Vision and Pattern Recognition, 2018. Springer International Publishing. pp 18–33.

Optimization Problems in Computer Vision

We are given some image data points $y = \{y_i; i = 1, \ldots N\}$.



a A deformed circle.



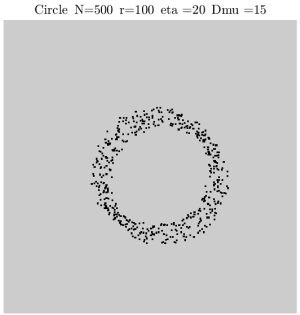
b. A deformed circle with clutter

A Bayesian approach is optimal. For example, to describe a circle we use three parameters (μ_x, μ_y, r). $\mathcal{S}_{\Theta=(r)}(x-\mu) \equiv 1 - \frac{(x-\mu)^2}{r^2} = 0$.

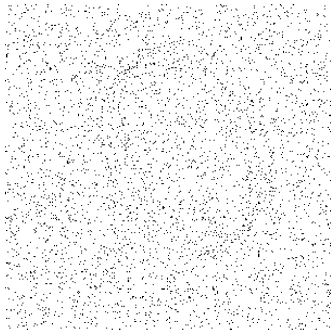
$$[\mu_x^*, \mu_y^*, r^*] = \arg\max P(y|\mu_x, \mu_y, r) = \frac{1}{Z} \prod_{i=1}^{N} \mathrm{e}^{-\beta \mathcal{S}^2_{\Theta=(r)}(y_i-\mu)}$$

Optimization Problems in Computer Vision

Extract Multiple Instances of a Shape.



a. Two deformed circles.



b. Three ellipses with clutter.

Bayesian approach for more objects require more parameters: difficult to model and to estimate. Alternative: Hough Transform. Votes

$$V(\mu_x, \mu_y, r) = \sum_{i=1}^{N} u\left(\frac{1}{\alpha} - |\mathcal{S}_{\Theta=(r)}(y_i - \mu)|\right)$$

where $u(x)$ is the Heaviside step function, $u = 1$ if $|\mathcal{S}_{\Theta}(x_i - \mu)| \leq \frac{1}{\alpha}$ and $u = 0$ otherwise.

Optimization Problems in Computer Vision

Extract Multiple Clusters of Points.

We are given some image data points $y = \{y_i; i = 1, \ldots N\}$.
Classical Gaussian mixture model. In 1D and C clusters

$$P(y|\{\mu_c, \sigma_c, \alpha_c;\ c = 1, \ldots, C\}) = \sum_{c=1}^{C} \alpha_c \frac{1}{\sqrt{2\pi\sigma_c^2}} \mathrm{e}^{-\frac{(y_i-\mu_c)^2}{2\sigma_c^2}}$$

where C is the number of classes and normalization $1 = \sum_c \alpha_c$. The EM algorithm, a statistical one, is one of the preferred choices.

Optimization Problems in Computer Vision

Elastica and In-painting.

Let the 2D curve Γ be parameterized by $x(s)$ with unit tangent $\frac{dx(s)}{ds} = e_\theta = (\cos\theta(s)\,,\ \sin\theta(s))$. The Elastica curve minimizes

$$S(\Gamma) = \int_\Gamma \left(\frac{1}{2\,\sigma^2}\,\kappa^2(s) + \gamma \right) ds$$

Given parameters σ, γ and an initial position and orientation, $\{\mathcal{X}_0, \theta_0\}$. The formulation was given by James Bernoulli in 1691.

Optimization Problems in Computer Vision

Elastica and In-painting.

Mumford gave a statistical version

$$P(\Gamma) = \frac{1}{Z}\mathrm{e}^{-S(\Gamma)} = \frac{1}{Z} \lim_{\substack{n \to \infty \\ n\epsilon = L}} \mathrm{e}^{-\sum_{i=0}^{n}\left[\frac{1}{2\sigma^2}\frac{(\theta_i-\theta_{i-1})^2}{\epsilon}+\gamma\epsilon\right]}$$

and derived the equation

$$\frac{\partial}{\partial s}\rho(x,\theta,s) \quad = \quad \left[\frac{\sigma^2}{2}\frac{\partial^2}{\partial\theta^2} - e_\theta \cdot \nabla - \gamma\right]\rho(x,\theta,s)$$

to model the density function of the Elastica optimization criteria when view as a stochastic process.

Classical Physics and Optimization

$F = ma \quad \Leftrightarrow \quad -\frac{\partial}{\partial x} V(x) = m \frac{d^2 x(t)}{dt^2}$

Lagrangian: $L\left(x(t), \frac{dx(t)}{dt}\right) = \frac{1}{2} m \left(\frac{dx(t)}{dt}\right)^2 - V(x)$.

Action: $S(\mathcal{X}_T) = \int_0^T L(x(t), \dot{x}(t))\, dt$,

where $\mathcal{X}_T = \{x(t); t \in [0, T]\}$ is a path.

Euler Lagrange Equation to find the **local** optimal path
$\mathcal{X}_T^* = \{\mathcal{X}^*(t); t \in [0, T]\}$

$$\frac{d}{dt}\left(\frac{\partial L(x, \dot{x})}{\partial \dot{x}}\right) - \frac{\partial L(x, \dot{x})}{\partial x} = 0 \quad \Rightarrow \quad F = ma$$

Optics and Optimization

Optics and Fermat Principle: Light travel shortest time path between A and B.
The trajectory of light $\mathcal{X}_A^B = \{x(t); t \in [t_A, t_B]\}$ "optimizes" the total amount of time

$$\mathcal{X}_{AB}^* = \arg \textbf{local} \min T(\mathcal{X}_{AB}) = \int_{t_A}^{t_B} dt = \frac{1}{c} \int_A^B n(x) dx$$

where $\frac{dx}{dt} = v(x) = \frac{c}{n(x)}$, with
$v(x)$- speed of light in a medium with refractive index $n(x)$,
c- speed of light in vacuum.

Snell's law, $n_1 \sin \theta_1 = n_2 \sin \theta_2$, follows from it.

From Classical Optics to Wave Optics

- **Optics and Fermat Principle**: Light travel shortest time path between A and B.

$$T(\mathcal{X}_{AB}) = \int_{t_A}^{t_B} dt = \frac{1}{c}\int_A^B n(x)dx$$

- **Wave Optics:** Optimization criteria becomes a phase. Sum over all paths.

  $E(x = B) = \sum_{\mathcal{X}_{AB}} e^{iT(\mathcal{X}_{AB})} = \sum_{\mathcal{X}_{AB}} e^{i\frac{1}{c}\int_A^B n(x)dx}$

  special case: constant index n:

  $E(x = B) = \sum_{\mathcal{X}_{AB}} e^{i\frac{1}{c}n|\mathcal{X}_{AB}|} = \sum_{\mathcal{X}_{AB}} e^{ik\,|\mathcal{X}_{AB}|}$, where $k = \frac{n}{c}$.

  Another View: Maxwell's equations

  $\nabla^2 E(x,t) = \left(\frac{n}{c}\right)^2 \frac{\partial^2}{\partial t^2} E(x,t)$ and for $E(x,t) = E(x)e^{-iwt}$

$$\nabla^2 E(x) + k^2(x)E(x) = 0$$

Quantum Path Integral

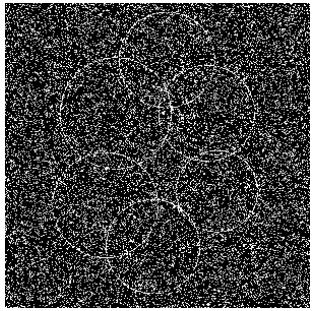
Consider states $x(t)$ that may vary over time. Wave propagation

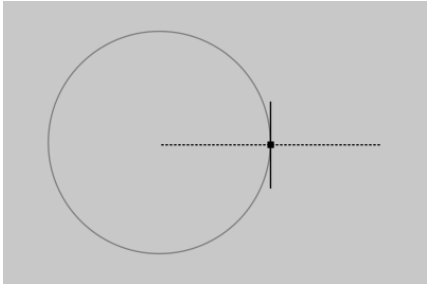
$$\psi_T(x(T)) = \int \mathrm{d}\mathcal{X}_0^T \, \frac{1}{Z} \, \mathrm{e}^{\mathrm{i}\frac{1}{\hbar}\,\mathcal{S}(\mathcal{X}_0^T)} \, \psi_0(x(0)) \, ,$$

where $\mathcal{X}_0^T$ is a path from initial state $x(0)$ to final state $x(T)$ and a hyper-parameter $\hbar$ is introduced. The integral is over all possible paths. The Optimization criteria becomes a phase. Sum over all paths.

Born rule: Probability is derived as $P(x,t) = |\psi_t(x(t))|^2$

Complex-Valued Hough Transform for Circle Detection





Hough transform for circles: When tangent information is available, and the radius is unknown, every point votes for the line perpendicular to the tangent.

Filter Responses: $(V(y), \tau(y) + \frac{\pi}{2}) = (\max_j V_j(y), \arg max_j V_j(y))$

Edges: maximum wavelet response at y over all orientation j (as long as $V(y) > Th$, an empirically defined threshold).

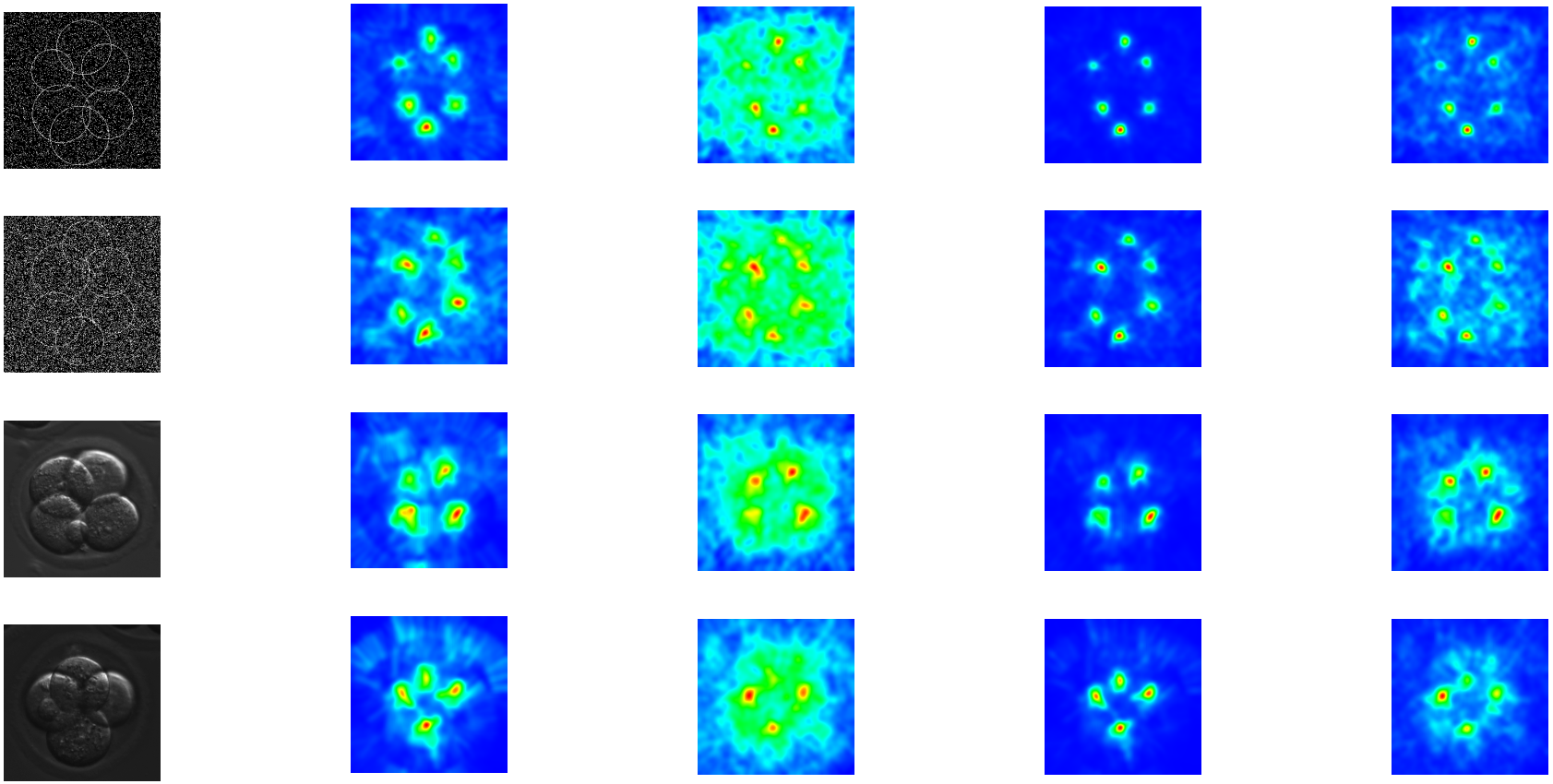
Experiments setting for Circle Detection

vote type	*weighted*	*vote*
constant real	yes	$\nu_V(x) = \sum_{y:x\in I(y)} V(y)$
constant real	no	$\nu(x) = \sum_{y:x\in I(y)} 1$
complex	yes	$\psi_V(x) = \sum_{y:x\in I(y)} V(y)e^{ik\lvert x-y\rvert}$
complex	no	$\psi(x) = \sum_{y:x\in I(y)} e^{ik\lvert x-y\rvert}$

The *shape likelihood*, L, is defined as $L(x) = \nu(x)^2$ and $L(x) = |\psi(x)|^2$, respectively. Detecting shapes consists of finding local maxima in L.

Experiments for Circle Detection

Results are from normalized maps



Data $\quad \nu_V^2(x) \quad \nu^2(x) \quad |\psi_V(x)|^2 \quad |\psi(x)|^2$

Experiments for Circle Detection

Outputs of the different methods. (a) Inputs. From top to bottom: synthetic image with noise 0.1, synthetic image with noise 0.2, 4 cells with small degree of overlap, 4 cells with large degree of overlap. (b) Voting space of classic method, weighted by edge strength. (c) Classic, non-weighted. (d) Complex, weighted by edge strength. (e) Complex, non-weighted.

Magnitude squared of complex number votes provide sharper accumulator spaces.

From: *Complex-Valued Hough Transforms for Circles.* M. Cicconet, D. Geiger and M. Werman. In the IEEE 22nd International Conference in Image Processing (ICIP), 2015, Quebec City, Canada.

Quantum Path Integral for Shapes

Consider states $(x(t), \Theta(t))$ that may vary over time. Wave propagation

$$\psi_0(\mu) = \int \mathrm{d}\mathcal{X}_0^T \, \mathcal{K}(\mathcal{X}_0^T) \, \psi_\Theta(x) \, ,$$

where $\mathcal{X}_0^T$ is a path from initial state $(x(0), \Theta(0)) = (x, \Theta)$ to final state $(x(T), \Theta(T)) = (\mu, 0)$. The integral is over all possible paths and

$$\mathcal{K}(\mathcal{X}\S_0^T) = \frac{1}{C} \, \mathrm{e}^{\mathrm{i}\frac{1}{\hbar} \, \mathcal{S}_{\Theta\to 0}^{x(0)\to\mu}(\mathcal{X}_0^T)} = \frac{1}{C} \, \mathrm{e}^{\mathrm{i}\frac{T}{\hbar} \, |\mathcal{S}_\Theta(x(0)-\mu)|} \, ,$$

where a hyper-parameter, $\hbar$, is introduced.

Quantum Path Integral for Shapes

Initial wave to be approximated by the data points,
$\psi_0(x) \approx \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \delta(x(0) - y_i)$. Then,

$$\psi_\Theta(\mu) \approx \sum_{i=1}^{N} \int \mathrm{d}x(0) \, \frac{\mathrm{e}^{\mathrm{i}\,\frac{T}{\hbar}\,|\mathcal{S}_\Theta(x(0)-\mu)|}}{C} \, \frac{1}{\sqrt{N}} \, \delta(x(0) - y_i)$$

$$= \frac{1}{C\sqrt{N}} \sum_{i=1}^{N} \mathrm{e}^{\mathrm{i}\,\frac{T}{\hbar}\,|\mathcal{S}_\Theta(y_i-\mu)|}$$

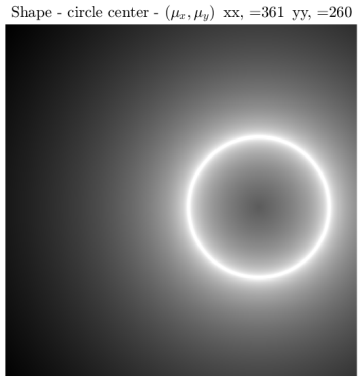
Thus, points of the deformed shape, y_i, are interpreted as evidence of different paths. For an ideal shape there is only optimal classical paths (no deformation).

Visualization: Adding/Voting Shape Terms

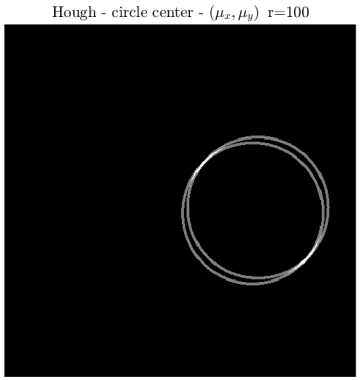
Note: Occlusion resistance with just 50 votes

$$P(\mu) \approx \frac{1}{C^2 N} \left| \sum_{i=1}^{N} \mathrm{e}^{\mathrm{i}\,\frac{T}{\hbar}\,|\mathcal{S}_{\Theta}(y_i-\mu)|} \right|^2, \quad V(\mu, r) = \sum_{i=1}^{N} u\left(\frac{1}{\alpha} - |\mathcal{S}_{\Theta=(r)}(y_i - \mu)|\right)$$

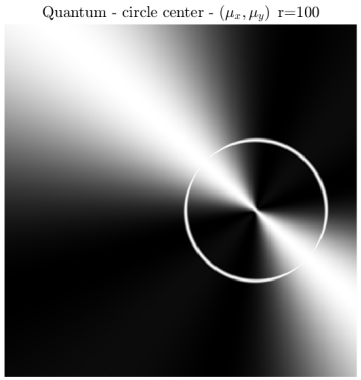
From input deformed circle first figure a., the first two data votes: adding terms $u\left(\frac{1}{\alpha} - |\mathcal{S}_{\Theta=(r)}(y_i - \mu)|\right)$ or $\mathrm{e}^{\mathrm{i}\,\frac{T}{\hbar}\,|\mathcal{S}_{\Theta}(y_i-\mu)|}$, where $\hbar = 0.03$, $\frac{1}{\alpha} = 0.02$



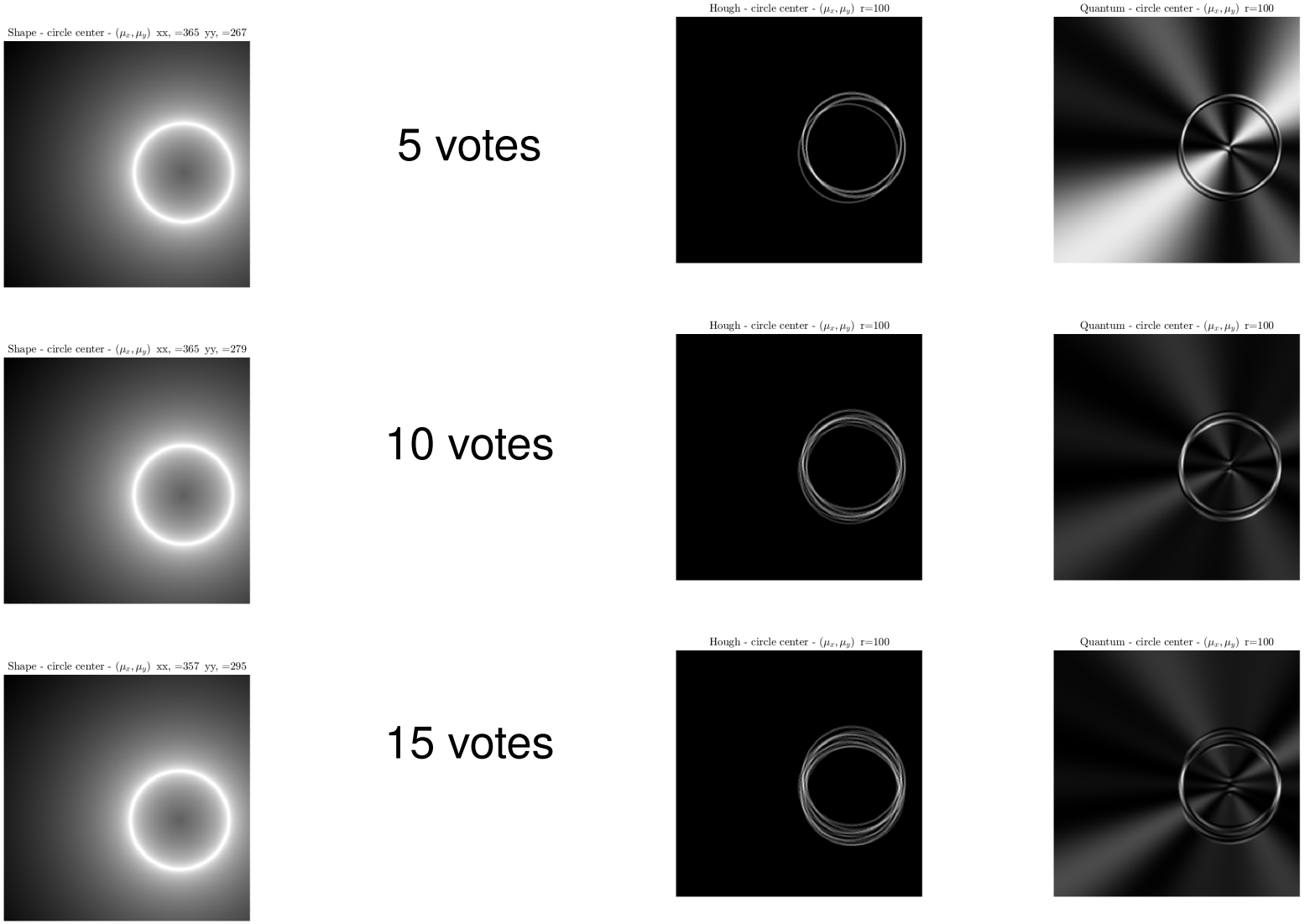
$|\mathcal{S}_{\Theta}(y_2 - \mu)|$



Hough Vote (y_1, y_2)



Quantum Vote (y_1, y_2)

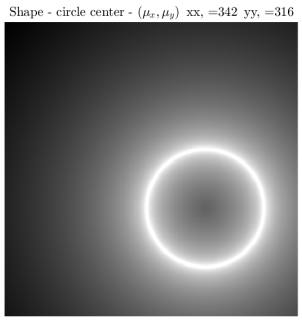


$|\mathcal{S}_\Theta(y_i - \mu)|$

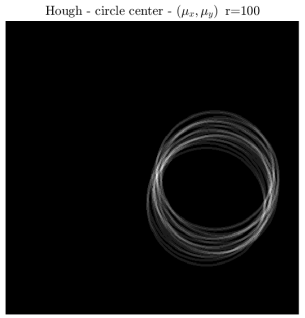
number of data votes

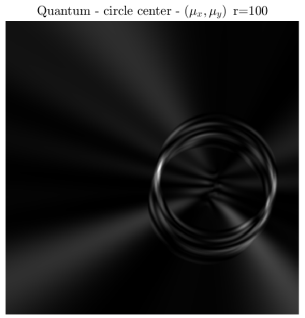
Hough Vote ($y_1 \ldots$y_15)

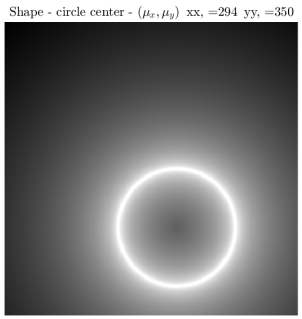
Quantum Vote ($y_1 \ldots$y_15)



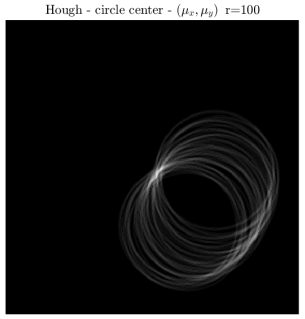
20 votes

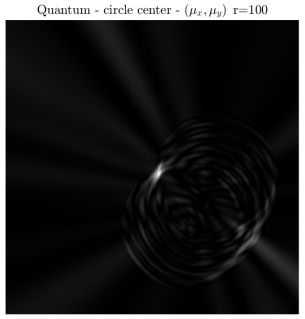


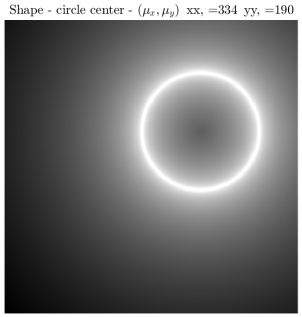




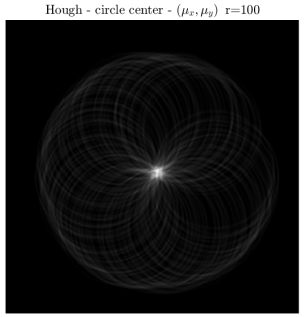
50 votes (occlusion resistance)

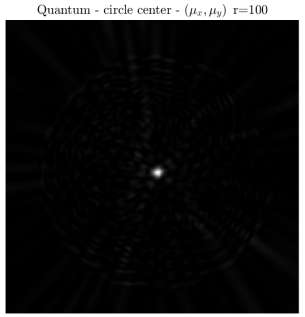






220 votes





$|\mathcal{S}_\Theta(y_i - \mu)|$

number of data votes

Hough Vote ($y_1 \ldots$y_220)

Quantum Vote ($y_1 \ldots$y_220)

Quantum Interference

The quantum probability associated with this probability amplitude (a pure state) is given by

$$\begin{aligned} \mathrm{P}_{\Theta}(\mu) = |\psi_{\Theta}(\mu)|^2 &= \frac{1}{C^2 N} \left| \sum_{i=1}^{N} \mathrm{e}^{\mathrm{i}\,\frac{T}{\hbar}\,|\mathcal{S}_{\Theta}(y_i-\mu)|} \right|^2 \\ &= \frac{1}{C^2\,N} \sum_{i=1}^{N} \left[1 + 2 \sum_{j>i}^{N} \cos\left[\frac{T}{\hbar} \left(|\mathcal{S}_{\Theta}(y_i-\mu)| - |\mathcal{S}_{\Theta}(y_j-\mu)| \right) \right] \right] \end{aligned} \tag{2}$$

where $\phi_{ij} = \frac{T}{\hbar}\big(|\mathcal{S}_{\Theta}(y_i-\mu)| - |\mathcal{S}_{\Theta}(y_j-\mu)|\big)$ is the "interference phase". Pairwise interaction resulting from single data votes, by squaring $\psi_0(\mu)$.

Quantum Interference

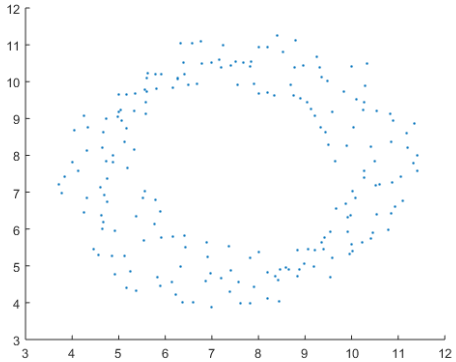
Analysis $\cos\phi_{ij}$ term, where $\phi_{ij} = \frac{T}{\hbar}\big(|\mathcal{S}_\Theta(y_i - \mu)| - |\mathcal{S}_\Theta(y_j - \mu)|\big)$.

Two shape points $|\mathcal{S}_\Theta(y_i - \mu)| \approx |\mathcal{S}_\Theta(y_j - \mu)|$ and $|\phi_{ij}| \ll 1$, resulting in $\cos\phi_{ij} \approx 1$.

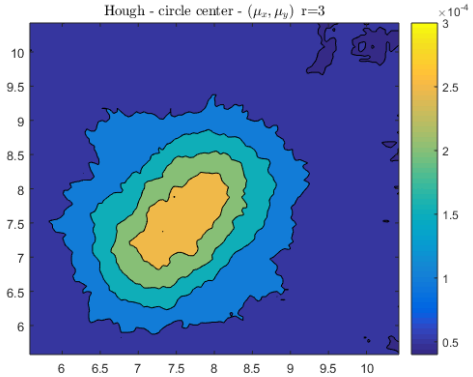
Two clutter points $|\mathcal{S}_\Theta(y_i - \mu)|$ and $|\mathcal{S}_\Theta(y_j - \mu)|$ are likely to be quite different. Also, small values of $\hbar$ result in arbitrary (random) values of $\mod(|\phi_{ij}|, 2\pi)$

Shape evaluated at misplaced center location.$\mu = \mu^* + \delta\mu$, or other misplaced set of parameters: even points belonging to the shape will behave like clutter, canceling each other .

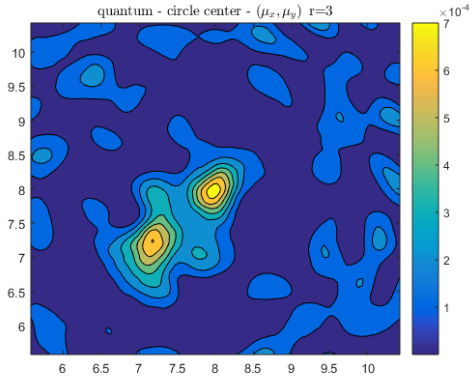
Experiments



(a) Two Overlapping Circles



(b) Hough Method

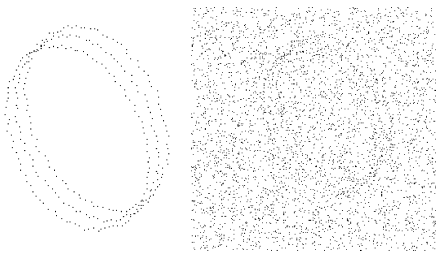


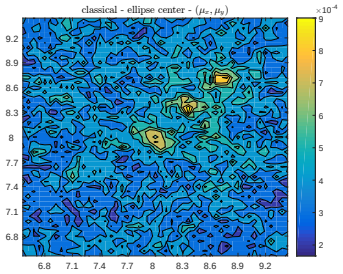
(c) Quantum Method

Figure: (a) Two overlapping circles deformed with $\eta = 0.5$, radius $r = 3$ and with 100 points each. The circle centers are at $(8, 8)$ and $(7.2, 7.2)$. The radius is fed to the method. (b) The Hough method with $\frac{1}{\alpha} = 0.37$. The method yields a probability that is more diluted and includes the correct centers, but does not suggest two peaks. (c) The quantum criterion with $\hbar = 0.12$. The *intereference* of the quantum method leads to two peak detections at nearby solutions.

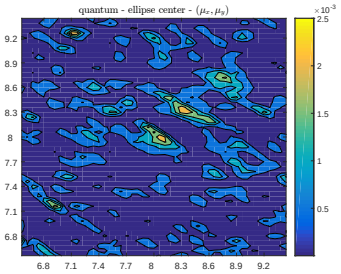
From: *Quantum Interference and Shape Detection*. D. Geiger and Z. Kedem. Energy Minimization Methods in Computer Vision and Pattern Recognition, 2018. Springer International Publishing. pp 18–33.

Experiments





(a)
Peak=0.9 10^{-3}
Valley=
0.2 10^{-3}



(b)
Peak=2.5 10^{-3}
Valley=
0.2 10^{-3}

Figure: Deformed ellipses, each with large axis $a = 3$, small axis $b = 2$, angle $\theta = 2$, and deformation $\eta = 0.05$. Centers at $\mu = (8.0, 8.0); (8.3, 8.3); (8.6, 8.6)$. We used the classical statistical method (3) with $\frac{1}{\alpha} = 0.04$ and the quantum criterion with $\hbar = 0.03$. In all of these experiments, *interference* of the quantum method leads to greater contrast between peak solutions and nearby solutions.

From: *Quantum Interference and Shape Detection*. D. Geiger and Z.

Kedem. Energy Minimization Methods in Computer Vision and Pattern Recognition, 2018. Springer International Publishing. pp 18–33.

Shape Detection Quantum vs Statistics

Classical probability from the quantum probability amplitude via the Wick rotation

$$\mathrm{i}\frac{T}{\hbar} \quad \rightarrow \quad \alpha$$

From the probability amplitude (2), the Wick rotation yields

$$\mathrm{P}_{\Theta}(\mu) = \frac{1}{Z}\sum_{i=1}^{N} \mathrm{e}^{-\alpha\,|\mathcal{S}_{\Theta}(y_i-\mu)|}\,. \tag{3}$$

y_i produces a vote $v(\Theta, \mu|y_i) = \mathrm{e}^{-\alpha\,|\mathcal{S}_{\Theta}(y_i-\mu)|}$, with values between $0 \leq v(\Theta, \mu|y_i) \leq 1$.

The hyper-parameter α controls the weight decay of the vote.

Clutter data with larger values for the shape error will cast negligible votes.

Shape Detection Quantum vs Statistics

The Hough-like transform with each vote $v(\Theta, \mu|y_i) = \mathrm{e}^{-\alpha\,|\mathcal{S}_\Theta(y_i-\mu)|}$ can be approximated by the binary vote

$$v(\Theta, \mu|y_i) = u\left(\frac{1}{\alpha} - |\mathcal{S}_\Theta(y_i - \mu)|\right),$$

The parameter $\frac{1}{\alpha}$ clearly defines the error tolerance for a data point y_i to belong to the shape $\mathcal{S}_\Theta(y - \mu)$. Resistant to occlusions and clutter. Still, these statistical models of shape do not exhibit the interference phenomenon.

Quantum Path Integral Elastica

A short cut derivation, from Mumford's equation

$$\frac{\partial}{\partial s}\rho(x,\theta,s) \quad = \quad \left[\frac{\sigma^2}{2}\frac{\partial^2}{\partial\theta^2} - e_\theta \cdot \nabla - \gamma\right]\rho(x,\theta,s)$$

Wick rotation

$$\begin{aligned} i\hbar\,\frac{\partial}{\partial s}\psi(x,\theta,s) &= -\left[i\hbar\, e_\theta \cdot \nabla + \gamma + \frac{\sigma^2\hbar^2}{2}\frac{\partial^2}{\partial\theta^2}\right]\psi(x,\theta,s) \\ &= \mathbf{H}\,\psi(x,\theta,s) \end{aligned}$$

$\mathrm{i}\hbar \to 1$. **H** is a Hermitian operator. A full path integral slice formulation with Feynman techniques also yields this equation. This is a quantum Elastica equation. Experiments need to be carried.